

TWO LEMMAS ON FORMAL POWER SERIES*

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Abstract

Let L be a local field and $\tilde{L}$ the completion of the maximal unramified extension of L . In this short note, we prove

(I) the sequences

$$(1'') \quad 0 \rightarrow \mathcal{O}_L[[X_1, \dots, X_n]] \rightarrow \mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]] \xrightarrow{\phi_L^{-1}} \mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]] \rightarrow 0$$

and

(2'')

$$1 \rightarrow \mathcal{O}_L[[X_1, \dots, X_n]]^x \rightarrow \mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]]^x \xrightarrow{\phi_L^{-1}} \mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]] \rightarrow 1$$

are exact, where θ_L is the Frobenius automorphism over L applied on the coefficients of $\mathcal{O}_L[[X_1, \dots, X_n]]$, and $\theta_L - 1$ respectively denotes the mapping $\alpha \mapsto \alpha^{\theta_L} - \alpha$ for

(1'') and $\epsilon \mapsto \frac{\epsilon^{\phi_L}}{\epsilon}$ for (2'');

(II) $\mathcal{C}^\circ(\tilde{L}, h)$ being the group the Coleman power series of degree 0, the sequence

$$1 \rightarrow \mathcal{C}^\circ(L, h) \rightarrow \mathcal{C}^\circ(\tilde{L}, h) \rightarrow \mathcal{C}^\circ(\tilde{L}, h) \rightarrow 1$$

is exact, where $\mathcal{C}^\circ(L, h) = \mathcal{O}_L[[X]]^x \cap \mathcal{C}^\circ(\tilde{L}, h)$

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1. Introduction

The following is well-known (see for example [3]): Let L be a local field, and $\tilde{L}$ the completion of the maximal unramified extension of L in a fixed algebraic closure Ω of L . Then

$$(i) \quad 0 \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_{\tilde{L}} \xrightarrow{\phi_L - 1} \mathcal{O}_{\tilde{L}} \rightarrow 0$$

and

$$(ii) \quad 1 \rightarrow \mathcal{O}_L^\times \rightarrow \mathcal{O}_{\tilde{L}}^\times \xrightarrow{\phi_L - 1} \mathcal{O}_{\tilde{L}}^\times \rightarrow 1$$

are both exact sequences, where ϕ_L stands for the Frobenius automorphism over L , and $\phi_L - 1$ respectively denotes the mapping $\alpha \mapsto \alpha^{\phi_L} - \alpha$ for (i), and $\epsilon \mapsto \frac{\epsilon^{\phi_L}}{\epsilon}$ for (ii).

In this section we give a formal extension of the above result to the case of formal power series ring. This fact may be well-known, but we did not notice any literature explicitly referring to this fact. So we believe that it might be worthwhile to report it.

We start with a lemma:

Lemma 1. *Let R be a commutative ring with unity, and α an automorphism of R such that*

(a) *the additive endomorphism $\alpha - 1$ of R defined by $x \mapsto x^\alpha - x$ ($x \in R$);*

(b) *the multiplicative endomorphism $\alpha - 1$ of $R^\times$ defined by $\epsilon \mapsto \frac{\epsilon^\alpha}{\epsilon}$ ($\epsilon \in R^\times$) are both surjective. Then, $R[[X]]$ and $R[[X]]^\times$ being the formal power series ring over R and its group of units respectively, the sequences*

$$(1) \quad 0 \rightarrow R_\alpha[[X]] \rightarrow R[[X]] \xrightarrow{\alpha - 1} R[[X]] \rightarrow 0$$

and

$$(2) \quad 1 \rightarrow R_\alpha[[X]]^\times \rightarrow R[[X]]^\times \xrightarrow{\alpha - 1} R[[X]]^\times \rightarrow 1$$

are both exact, where the automorphism α is applied only on the coefficients, and R_α denotes the fixed subring of α .

Hence, $\alpha - 1$ is surjective on $R[[X]]$ and on $R[[X]]^\times$.

Proof. The proof of the first case is trivial. As for the second case, let $\sum_{v=0}^{\infty} a_v X^v$ (with $a_0 \in R^x$) be any element in $R[[X]]^x$. We are looking for an element $\sum_{\mu=0}^{\infty} b_{\mu} X^{\mu}$ (with $b_0 \in R^x$) in $R[[X]]^x$ satisfying

$$(A) \quad \sum_{\mu=0}^{\infty} b_{\mu}^{\alpha} X^{\mu} = \left(\sum_{v=0}^{\infty} a_v X^v \right) \left(\sum_{\mu=0}^{\infty} b_{\mu} X^{\mu} \right) = \sum_{\lambda=0}^{\infty} \left(\sum_{\mu=0}^{\lambda} a_{\lambda-\mu} b_{\mu} \right) X^{\lambda}.$$

For $\lambda = 0$, we have $b_0^{\alpha} = a_0 b_0$ whose solution b_0 can be found in R^x by the assumption. Assume that b_{μ} (for $\mu = 0, 1, \dots, n-1$) have already been found in R . Then, for $\lambda = n$, the equation (A) takes the form

$$(B) \quad b_n^{\alpha} = a_0 b_n + \sum_{\mu=0}^{n-1} a_{n-\mu} b_{\mu} = \frac{b_0^{\alpha}}{b_0} b_n + \sum_{\mu=0}^{n-1} a_{n-\mu} b_{\mu},$$

where the last term is in R by the induction assumption. From (B), we obtain

$$(C) \quad \left(\frac{b_n}{b_0} \right)^{\alpha} = \frac{b_n}{b_0} + \frac{\sum_{\mu=0}^{n-1} a_{n-\mu} b_{\mu}}{b_0^{\alpha}}.$$

Since $b_0 \in R^x$, the last term above is in R . By (a), we can find $\frac{b_n}{b_0} \in R$ satisfying the equation (C). This completes the proof of the existence of $\sum_{\mu=0}^{\infty} b_{\mu} X^{\mu} \in R[[X]]^x$ satisfying (A). Now, the exactness of the second sequence follows easily.

Note that $R[[X_1, \dots, X_n]]$ is naturally identified with $R[[X_1, \dots, X_{n-1}]][[X_n]]$ (see [4], pp. 206). Hence, as a corollary, we get: □

Corollary 1. *Under the same assumptions and notation as above, the sequences*

$$(I') \quad 0 \rightarrow R_{\alpha}[[X_1, \dots, X_n]] \rightarrow R[[X_1, \dots, X_n]]^{\alpha-1} \rightarrow R[[X_1, \dots, X_n]] \rightarrow 0$$

and

$$(J') \quad 1 \rightarrow R_{\alpha}[[X_1, \dots, X_n]]^x \rightarrow R[[X_1, \dots, X_n]]^x \xrightarrow{\alpha-1} R[[X_1, \dots, X_n]]^x \rightarrow 1$$

are both exact.

Hence $\alpha - 1$ is surjective on $R[[X_1, \dots, X_n]]$ and on $R[[X_1, \dots, X_n]]^x$.

Corollary 2. *Let L be a local field, and $\tilde{L}$ the completion of the maximal unramified extension of L . Then, $\mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]]$ and $\mathcal{O}_{\tilde{L}}[[X_1, \dots, X_n]]^x$ being the formal power series ring over the ring of integers $\mathcal{O}_{\tilde{L}}$ in $\tilde{L}$ and its group of units respectively, the sequences*

$$(1'') \quad 0 \rightarrow \mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]] \rightarrow \mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]^{\phi_L^{-1}} \mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]] \rightarrow 0$$

and

$$(2'') \quad 1 \rightarrow \mathcal{O}_L[[X_1, \dots, X_n]]^x \rightarrow \mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]^x \xrightarrow{\phi_L^{-1}} \mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]^x \rightarrow 1$$

are both exact, where $\mathcal{O}_L[[X_1, \dots, X_n]]$ is the ring of formal power series with coefficients in $\mathcal{O}_L$ and ϕ_L is the Frobenius automorphism over L applied on the coefficients of $\mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]$.

Hence, $\phi_L - 1$ is surjective on $\mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]$ and on $\mathcal{O}_{\bar{L}}[[X_1, \dots, X_n]]^x$.

Corollary 3. *Let $\bar{k}$ be the algebraic closure of a finite field k . Then the sequences*

$$(1''') \quad 0 \rightarrow k[[X_1, \dots, X_n]] \rightarrow \bar{k}[[X_1, \dots, X_n]]^{\phi_k^{-1}} \bar{k}[[X_1, \dots, X_n]] \rightarrow 1$$

and

$$(2''') \quad 1 \rightarrow k[[X_1, \dots, X_n]]^x \rightarrow \bar{k}[[X_1, \dots, X_n]]^x \xrightarrow{\phi_k^{-1}} \bar{k}[[X_1, \dots, X_n]]^x \rightarrow 1$$

are exact, where ϕ_k stands for the Frobenius automorphism over k , i.e. $\alpha \mapsto \alpha^q$ for all $\alpha \in \bar{k}$ with $q = |k|$, applied on the coefficients of $\bar{k}[[X_1, \dots, X_n]]$.

Hence, $\phi_k - 1$ is surjective on $\bar{k}[[X_1, \dots, X_n]]$ and on $\bar{k}[[X_1, \dots, X_n]]^x$.

If the set of indeterminates is 0, then Corollary 2 reduces to the classical case mentioned in the beginning of the note.

2. Let L be a local field, and let $\tilde{L}$ be the completion of the maximal unramified extension of L in a fixed algebraic closure Ω of L . The completion of Ω will be denoted by $\hat{\Omega}$. Further, let π be a prime of $\mathcal{O}_{\tilde{L}}$, the ring of integers in $\tilde{L}$, h a Lubin-Tate series in $\mathcal{O}_{\tilde{L}}[[X]]$, and $F_h(X, Y)$ the (unique) formal group law belonging to h . Coleman in [1] introduced a norm operator $\mathcal{N}_h : \mathcal{O}_{\tilde{L}}((X))^x \rightarrow \mathcal{O}_{\tilde{L}}((X))^x$ by

$$(iii) \quad (\mathcal{N}_h g) \circ h = \prod_{\omega \in W_h^1} g(X[+_h]\omega),$$

where $\mathcal{O}_{\tilde{L}}((X))^x$ is the multiplicative group of units in the ring of Laurent series over $\mathcal{O}_{\tilde{L}}$, $W_h^1 := \{\omega \in \hat{\Omega} : h(\omega) = 0\}$, and $X[+_h]\omega = F_h(X, \omega)$ (see [2]).

Then, $\mathcal{N}_h$ is an endomorphism of $\mathcal{O}_{\tilde{L}}((X))^x$ satisfying $\mathcal{N}_h g \equiv g^{\phi_L} \pmod{\pi}$ for all $g \in \mathcal{O}_{\tilde{L}}((X))^x$ (see Theorem 11 in [1]). The Laurent series $g \in \mathcal{O}_{\tilde{L}}((X))^x$ satisfying $\mathcal{N}_h g = g^{\phi_L}$ are called Coleman power series, and the subgroup of $\mathcal{O}_{\tilde{L}}((X))^x$ consisting of all Coleman series is denoted by $\mathcal{C}(\tilde{L}, h)$. In particular, $\mathcal{C}^\circ(\tilde{L}, h) := \mathcal{C}(\tilde{L}, h) \cap \mathcal{O}_{\tilde{L}}[[X]]^x$, the group of Coleman series of degree 0, plays a fundamental role in the construction of the non-abelian local class field theory. The most significant feature of $\mathcal{C}(\tilde{L}, h)$ is the

following: upon reduction mod π , $C(\tilde{L}, h)$ is isomorphically mapped onto $\bar{\mathcal{F}}_q((X))^x$ where $\mathcal{F}_q$ denotes the residue class field of L , and $\bar{\mathcal{F}}_q$ for its algebraic closure. Hence, upon reduction mod π , $C^\circ(\tilde{L}, h)$ is isomorphically mapped onto $\bar{\mathcal{F}}_q[[X]]^x$.

Lemma 2. *Keeping all notations as above, assume that the Lubin-Tate formal power series $h \in \mathcal{O}_L[[X]]$. Then, the sequence*

$$1 \rightarrow C^\circ(L, f) \rightarrow C^\circ(\tilde{L}, f) \xrightarrow{\phi_L^{-1}} C^\circ(\tilde{L}, f) \rightarrow 1$$

is exact, where $C^\circ(L, f) = \mathcal{O}_L[[X]]^x \cap C^\circ(\tilde{L}, f)$.

Proof. *It suffices to show the surjectivity of $\phi_L - 1$. Before doing this, some remarks will be in place. By the assumption, the Lubin-Tate series h is in $\mathcal{O}_L[[X]]$, i.e. $h^{\phi_L} = h$. Hence, by the remark on pp. 49 of [3], $F_h(X, Y) \in \mathcal{O}_L[[X, Y]]$. Furthermore, $h^{\phi_L} = h$ implies that $\omega^{\phi_L} \in W_h^1$ for all $\omega \in W_h^1$. Hence, for any $\omega \in W_h^1$,*

$$(X[+_h]\omega)^{\phi_L} = (F_h(X, \omega))^{\phi_L} = F_h^{\phi_L}(X, \omega^{\phi_L}) = F_h(X, \omega^{\phi_L}) = X[+_h]\omega^{\phi_L},$$

where $\omega^{\phi_L} \in W_h^1$.

From these remarks, it follows that $g^{\phi_L} \in C^\circ(\tilde{L}, h)$ for every $g \in C^\circ(\tilde{L}, h)$. In fact, by (iii), we have

$$\begin{aligned} \mathcal{N}_h(g^{\phi_L}) \circ h &= \prod_{\omega \in W_h^1} g^{\phi_L}(X[+_h]\omega) \\ &= \prod_{\omega \in W_h^1} g^{\phi_L}((X[+_h]\omega)^{\phi_L}) \\ &= \left(\prod_{\omega \in W_h^1} g(X[+_h]\omega) \right)^{\phi_L} \\ &= (\mathcal{N}_h(g) \circ h)^{\phi_L} \\ &= (g^{\phi_L} \circ h)^{\phi_L} = (g^{\phi_L})^{\phi_L} \circ h, \end{aligned}$$

so that $\mathcal{N}_h(g^{\phi_L}) - (g^{\phi_L})^{\phi_L}$ (otherwise, the power series $\mathcal{N}_h(g)^{\phi_L} - (g^{\phi_L})^{\phi_L}$ will have infinitely many zeroes with increasing exponential values). Hence $g^{\phi_L} \in C^\circ(\tilde{L}, h)$ as

desired. Note that at the second equality before the last one, the assumption $g \in \mathcal{C}^\circ(\tilde{L}, h)$ was used. Now, returning to the proof of our assertion, take any $f \in \mathcal{C}^\circ(\tilde{L}, h)$ and set $f \pmod{\pi} = \xi \in \bar{\mathcal{F}}_q[[X]]^x$. Then, by Corollary 3 above, there is a $\xi \in \bar{\mathcal{F}}_q[[X]]^x$ satisfying $\frac{\xi_1^{\phi_{F_q}}}{\xi_1} = \xi$. Let f_1 be the unique element of $\mathcal{C}^\circ(\tilde{L}, h)$ such that $f_1 \pmod{\pi} = \xi_1$. Then $f_1^{\phi_L} \in \mathcal{C}^\circ(\tilde{L}, h)$ by the remark above, and $\frac{f_1^{\phi_L}}{f_1} \in \mathcal{C}^\circ(\tilde{L}, h)$. Furthermore, upon reduction mod π , $\frac{f_1^{\phi_L}}{f_1}$ yields $\frac{\xi_1^{\phi_{F_q}}}{\xi_1} = \xi$, so that $\frac{f_1^{\phi_L}}{f_1} = f$ by the uniqueness of the lifting. This proves our assertion.

References

- [1] R. Coleman, *Division values in local fields*, Invent. Math **53** (1979), 91-116.
- [2] E. de Shalit, *Iwasawa Theory of Elliptic Curves with Complex Multiplication*, Perspectives in Math., vol. 3, Academic Press, Boston-Orlando, 1987.
- [3] K. Iwasawa, *Local Class Field Theory*, Oxford University Press, Oxford-New York, 1986.
- [4] S. Lang, *Algebra*, 3rd ed., Addison-Wesley, Reading-Massachusetts, 1995.

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